ON THE APPLICABILITY OF THE THEORY OF ONE-DIMENSIONAL FLOW IN THE PRESENCE OF HALL CURRENTS

(O PRIMENIMOSTI TEORII ODNOMERNOGO POTOKA PRI NALICHII TOKOV KHOLLA)

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The generalized Ohm's law for weakly ionised plasma, as is well known, has the form

$$\mathbf{j} + \mathbf{j}\times\omega\tau = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \left(\omega\tau = \frac{e\mathbf{B}\tau}{m}\right) \tag{1}$$

Here ω is the electron cyclotron frequency, τ is the free flight time of the electron. The magnitude of $\omega\tau$ depends on the concentration of neutral particles, and therefore the occurrence of a concentration gradient leads to a distortion of the current lines in comparison with their distribution in a homogeneous medium. The influence of density fluctuations on current flow was studied in papers [1 and 2], where, in particular, it was noted that, when $\omega\tau$, is not small compared with unity, the effect of non-homogeneities depends essentially on their actual distribution in space.

When $\omega\tau \gtrsim 1$ the distribution of current is, generally speaking, non-homogeneous. However for sectioned electrodes, if the length of the electrodes along the flow is significantly less than the distance between them, the Hall current in incompressible liquids can be neglected [3]. The occurrence of a non-one-dimensional current distribution owing to the variation of $\omega\tau$ in a compressible liquid under the influence of a magnetic field is also unimportant because the density gradients which appear are perpendicular to the direction of current and therefore weakly influence its magnitude. In these cases, for ducts of constant cross-section with sufficiently small length of sectioning, the one-dimensional equations are applicable. If however the duct cross-section vary, as, for example, with flow in nozzles, or the duct walls are uneven, then non-one-dimensional flow arises because of the appearance of velocity components perpendicular to the duct axis, and the occurrence of non-uniform density distribution. As is shown in the present paper, small nonhomogeneities in density on the current path lead to intense variations in it, and the one-dimensional equations become nonapplicable with sufficiently large $\omega\tau$ even in the case of sectioned electrodes. We will examine the influence of flow nonhomogeneities in the example of a symmetric two-dimensional nozzle with slowly varying cross-section, and in the example of subsonic laminar flow in a duct with a profile having sufficient curvature.

We restrict ourselves to the case of sectioned electrodes and will assume the conductivity to be constant. The equations of magneto-hydrodynamics, taking into account the Hall current, have the form

$$\operatorname{div} \rho \mathbf{v} = 0, \qquad \rho (\mathbf{v}\nabla) \mathbf{v} = -\nabla p + \mathbf{j}\times\mathbf{B}, \qquad \mathbf{v}\nabla p = \gamma \frac{p}{\rho} \mathbf{v}\nabla\rho + (\gamma - 1)\frac{j^2}{\sigma}$$

$$\mathbf{j} + \mathbf{j}\times\boldsymbol{\delta} = \sigma(-\nabla\varphi + \mathbf{v}\times\mathbf{B}), \quad \operatorname{div}\mathbf{j} = 0 \qquad \left(\delta = \frac{e\mathbf{B}\tau}{m}\right) \tag{2}$$

Here φ is the electrical potential.

Accounting for the joint influence of the variation in nozzle section and the magnetic field on the flow is rather difficult; therefore for simplicity we assume that the hydrodynamic influence on the flow, due to the variation in section, is greater than the magneto-hydrodynamic influence. The magnetic field shows an essential effect on the flow over a length of the order of $\rho v / \sigma B^2$. If we examine the flow over a smaller distance (i.e. if we consider a length of section smaller than this characteristic length), then we may neglect the influence of the current and potential on the distribution of the hydrodynamic quantities. Therefore we will first determine the magnitude of the corrections to the one-dimensional flow equations without considering the effect of the magnetic field on the flow. Let

$$\rho = \rho_0 + \rho_1, \quad p = p_0 + p_1 \qquad \left(\rho_0 = \frac{1}{y_2 - y_1}\int_{y_1}^{y_2} \rho dy\right)$$

where $y_1(x), y_2(x)$ are the equations of the nozzle boundary. The mean velocity of flow is directed along the x-axis.

We will also assume that the overall variation of cross-section is small. This allows us to take the average quantities $\rho_0, v_0, \ldots$ to be slowly varying functions of x and to neglect their derivatives.

Integrating the continuity equation

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) = 0$$

within the limits y_1 and y_2, taking into account the vanishing of the normal component of the velocity at the wall, we get the general equation of one-dimensional motion

$$\frac{\partial}{\partial x}[(\rho v_x)_0 Y] = 0, \qquad Y = y_2 - y_1 \tag{3}$$

At the same time, to the accuracy of quadratic terms

$$\frac{\partial}{\partial x}(\rho v_x)_0 + \rho_0 \frac{\partial v_{1x}}{\partial y} + v_0 \frac{\partial \rho_1}{\partial x} + \rho_0 \frac{\partial v_{1y}}{\partial y} = 0 \tag{4}$$

and the average value of a product equals the product of average values, as for example $(\rho v_x)_0 = \rho_0 v_0$.

Subtracting (3) from (4) we obtain the equation for determining the correction to the continuity equation

$$\frac{\partial v_{1x}}{\partial x} + \frac{v_0}{\rho_0}\frac{\partial \rho_1}{\partial x} + \frac{\partial v_{1y}}{\partial y} - \frac{v_0}{Y}\frac{dY}{dx} = 0 \qquad (Y = y_2 - y_1) \tag{5}$$

The remaining equations for the corrections may be obtained analogously

$$\rho_0 v_0 \frac{\partial v_{1x}}{\partial x} = -\frac{\partial p_1}{\partial x}, \quad \rho_0 v_0 \frac{\partial v_{1y}}{\partial x} = -\frac{\partial p_1}{\partial y}, \quad \frac{\partial p_1}{\partial x} = \gamma \frac{p_0}{\rho_0}\frac{\partial \rho_1}{\partial x} \tag{6}$$

In the derivation of Equation (6) it was taken that the correction to the pressure is an even function of y for a symmetrical nozzle. The boundary conditions for the system (5) and (6) consist in equating to zero the velocity components normal to the nozzle wall.

$$v_{1y}(y = y_1) = v_0 dy_1 / dx, \; v_{1y}(y = y_2) = v_0 dy_2 / dx$$

Let the equations of the boundary be given in the form

$$y_{1,2} = \pm f(\varepsilon x / L) \tag{7}$$

Here ε is a small quantity and L is some characteristic length. In this case the corrections to the mean value of the density are quantities of order ε^2. Indeed, keeping the fundamental terms in the first equation of the system (5), (6), we get

$$v_{1y} = v_0 \frac{1}{Y}\frac{dY}{dx} y$$

that is, the linear distribution y the component of velocity along the section. The next equation for v_{1y} gives

$$\frac{\partial p_1}{\partial y} = -\rho_0 v_0^2 y \frac{d}{dx}\left(\frac{1}{Y}\frac{dY}{dx}\right) \tag{8}$$

But if the equation for the boundary of the nozzle is given in the form (7), then according to (8) the corrections to the pressure are of second order of smallness in ε, just as are the corrections to the mean value of the density. Now we may also convince ourselves that the neglected terms in (5) are indeed of higher order in ε. Further, with given distributions of hydrodynamic quantities we determine the perturbations in current and potential. For sectioned electrodes, if the width of the duct significantly exceeds the length of the electrodes along the duct, the distributions of current and electric field, with conditions of constant velocity and constant cross-section, may be considered homogeneous, except in the region of the electrodes. Therefore we will look for deviations from a homogeneous distribution in the current field, supposing for simplicity that the electrodes in each section are short circuited and no average current along the x-axis exists. We will consider δ inversely proportional to the density of neutral particles; therefore

$$\delta_1 = -\delta_0 \rho_1 / \rho_0$$

The magnetic field is directed along the z-axis.

With these conditions, from the two last equations of system (2), linearizing them and eliminating the current components, we get for the determination of the perturbation in potential Poisson's equation

$$\Delta\varphi_1 = \delta_0 B\left(\frac{\partial v_{1x}}{\partial x} - \frac{v_0}{\rho_0}\frac{\partial \rho_1}{\partial x} - \frac{\delta_0 v_0}{\rho_0}\frac{\partial \rho_1}{\partial y} + \frac{\partial v_{1y}}{\partial y}\right) + B\left(\frac{\partial v_{1y}}{\partial x} - \frac{\partial v_{1x}}{\partial y}\right) \tag{9}$$

For the nozzle, formed of non-conducting walls, the perturbation in potential also will be determined from Equation (9), where it is necessary to set $v_{1y} = 0, \partial\rho_1 / \partial y = 0,$ since the variation in hydrodynamical quantities occurs in the x, z plane.

Consequently, the corrections to the potential with large δ are of order $\delta^2\varepsilon^2$ for nozzles formed of electrodes, or $\delta\varepsilon$ for nozzles formed of non-conducting walls. From the expression for the corrections to the current

$$j_{1x} = \frac{\sigma}{1+\delta_0^2}\left(\delta_0 \frac{\partial\varphi_1}{\partial y} + \delta_0 B v_{1x} - v_0 B \frac{\rho_1}{\rho_0}\delta_0 - \frac{\partial\varphi_1}{\partial x} + v_{1y} B\right)$$
$$j_{1y} = -\frac{\sigma}{1+\delta_0^2}\left(\frac{\partial\varphi_1}{\partial y} + B v_{1x}\right) - \frac{\sigma\delta_0}{1+\delta_0^2}\left(v_0 B \frac{\rho_1}{\rho_0}\delta_0 + \frac{\partial\varphi_1}{\partial x} - v_{1y} B\right) \tag{10}$$

it follows that they are of order $\varepsilon^2\delta$, if nonhomogeneity arises on the current path, and simply of order ε with nonhomogeneities in a plane perpendicular to the current. Thus for the applicability of the one-dimensional equations it is sufficient that $\varepsilon^2\delta \ll 1$ and $|\varepsilon| \ll 1$ for nozzles with slowly decreasing cross-section.

We will consider therefore in greater detail the case of a profile whose second derivative is not a small quantity. The two-dimensional flow in a duct, one of whose boundaries is sinusoidal, may serve as the most characteristic example, inasmuch as in many cases the function, giving the equation of the wall, may be resolved in a Fourier series. In subsonic flow the perturbations created by a wavy wall damp out exponentially with the distance from it (see, for example, [4]). Therefore if the walls are situated sufficiently far from one another then we can disregard their mutual influence (if, for example, one of the walls is wavy while the other is smooth; or both walls are wavy).

The corrections v_{1x}, v_{1y} to the velocity of unperturbed motion v_0 (along the x-axis), if the equation of the wavy wall is expressed in the form $y = \varepsilon_0 \sin kx$ (the second wall is taken to be smooth), have the form

$$v_{1x} = v_0 \frac{\varepsilon_0 k}{b} e^{-\varkappa y} \sin kx, \qquad v_{1y} = \varepsilon_0 k v_0 e^{-\varkappa y} \cos kx$$
$$b^2 = 1 - M^2, \quad \varkappa = kb, \quad M^2 = \frac{v_0^2\rho_0}{\gamma p_0}$$

We find the changes in pressure and density from the linearized equations of system (2)

$$p_1 = \gamma \frac{p_0}{\rho_0}\rho_1, \qquad \rho_0 v_0 \frac{\partial v_{1x}}{\partial x} = -\frac{\partial p_1}{\partial x}$$

that is,

$$\rho_1 = -\frac{\varepsilon_0 k M^2 \rho_0 e^{-\varkappa y}}{b}\sin kx$$

Equation (9) in this case takes the form

$$\Delta\varphi_1 = v_0 B \delta_0 M^2 \varepsilon k^2 e^{-\varkappa y}\left(\frac{2\cos kx}{b} - \delta_0 \sin kx\right) \tag{11}$$

The boundary conditions consist in equating to zero the potential perturbations at the electrodes

$$\varphi_1 = 0, \; y = 0, \; l \tag{12}$$

and setting to zero the components of current along the duct axis at the boundaries of the sections

$$j_{1x} = 0, \quad \text{for} \quad \frac{\partial\varphi_1}{\partial x} = \delta_0\frac{\partial\varphi_1}{\partial y} + B\left(v_{1y} - \delta_0 v_0 \frac{\rho_1}{\rho_0} + \delta_0 v_{1x}\right) \qquad (x = x_1, x_2) \tag{13}$$

When $\delta_0 \gg 1$ the most essential terms in Expression (10) for j_{1y} become $(\sigma / \delta_0)\, \partial\varphi_1 / \partial x$, and in the expression for j_{1x}, the term $(\sigma / \delta_0)\, \partial\varphi_1 / \partial y$. These terms, generally speaking, are of order $\varepsilon_0\delta_0 k$. Indeed, the boundary conditions (12) and (13) (if the order of the terms in (13) is taken into account) reduce to the simpler conditions: $\varphi_1 = 0$ on the boundaries of the rectangle $x_1 \leqslant x \leqslant x_2,\ 0 \leqslant y \leqslant l$. The solution to Poisson's equation in this case can be written in the form

$$\varphi_1 = \sum_{n=1}^{\infty} \varphi_{1n}(x) \sin \alpha_n y \qquad \left(\alpha_n = \frac{n\pi}{l}\right)$$

$$\varphi_{1n} = \frac{2v_0 B\delta_0 M^2 \varepsilon k^2 \alpha_n}{l(k^2 + \alpha_n^2)(\varkappa^2 + \alpha_n^2)} \left[\frac{\sin kx_1 \sinh \alpha_n (x - x_2) + \sin kx_2 \sinh \alpha_n (x_1 - x)}{\sinh \alpha_n (x_2 - x_1)} - \sin kx \right]$$

(in the computation it assumed that $\varkappa l \gg 1$).

On the electrodes the most essential term in Expression (10) for j_{1y}, that is to say $(\sigma / \delta_0) / (\partial\varphi_1 / \partial x)$, vanishes, however the current j_{1x}, generally speaking, is not small. The occurrence of isolated currents (of order $\varepsilon_0 k\delta_0$) leads to a dissipation of flow energy. The magnitude of the current is determined by the Mach number, by the method of sectioning.

Thus, the presence of nonhomogeneities in the distribution of quantities over the duct cross-section with sectioned electrodes leads to the essential corrections to the one-dimensional flow equations with large $\omega\tau$, in particular if the duct walls possess sufficient curvature. The magnitude of the current perturbations are determined by the magnitude of the perturbations in density. For nozzles with slowly decreasing cross-section the corrections to the current (with $(\omega\tau)^2 \gg 1$) are of order $\varepsilon^2\omega\tau$, if ε is the nozzle inclination angle. In subsonic laminar flow the density perturbations due to a wavy wall, are of order ε_0 / λ (λ is the wave length, ε_0 its amplitude), and the current perturbations are of order $\varepsilon_0\omega\tau / \lambda$ (in relation to the currents in a duct with constant cross-section). The Mach number of the flow has an essential influence on the magnitude of density perturbations and, consequently, on the perturbations in current. These perturbations in subsonic flow decrease with decreasing Mach number as M^2.

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